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ON THE SCHUR MULTIPLIER OF AUGMENTED ALGEBRAS

BY

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ABSTRACT

Let k be a field, and A a finitely generated k-algebra, with augmentation. Suppose there is a presentation of A

$$0 \to I \to R \to A \to 0$$

where R is a finitely generated free k-algebra and I is non-zero. If A is infinite dimensional over k, Lewin proved that R/I^2 is not finitely presented. A stronger statement would be that the 'Schur multiplier' of R/I^2 is not finite dimensional. In the case that A is an augmented domain, we prove this stronger statement, and some related statements.

1. Introduction

If R is a free associative algebra, over a field, and I is a non-zero two-sided ideal of R, then Lewin proved [7] that I^2 is not finitely generated (as a 2-sided ideal!) when the algebra R/I is infinite dimensional. In other words, R/I^2 is not finitely presented in this case. On the other hand, it is easy to see that when R is finitely generated and R/I is finite dimensional, so is R/I^2, and hence I^2 is finitely generated.

Similar behavior is seen in groups. If F is a finitely generated free group, and R is a normal subgroup then R' is *normally* finitely generated if, and only if, F/R is finite. But Baumslag, Strebel and Thomson proved [2] a stronger fact. Denoting the m-th member of the lower central series by γ_m, they proved that for

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$m > 1$ the Schur multiplier of $F/\gamma_m R$, $H_2(F/\gamma_m R, \mathbb{Z})$, is not finitely generated (as an abelian group) if F/R is not finite.

In Lie algebras even more is true. It is shown in [1] that if L is a free Lie algebra over a field k, and I is *any* non-zero proper ideal then, for $n \geq 2$, $H_2(L/I^n, k)$ is not finite dimensional, and thus I^n is not finitely generated as an ideal. Here $I^n = I$ if $n = 1$, and $I^n = [I^{n-1}, I]$ if $n > 1$. This is true even when L/I is finite dimensional.

In the case of associative algebras, the question arises whether an analogous result can be attained. The analogue of the Schur multiplier of a group is the second Tor functor, i.e. if A is an augmented k-algebra, $\mathrm{Tor}_2^A(k,k)$ is considered as the Schur multiplier of A. This agrees with the definitions in group theory where A is the group ring $\mathbb{Z}G$, and for Lie algebras where A is the universal enveloping algebra. It is quite easy to see that as in the case of groups and Lie algebras, if A is finitely presented as a k-algebra then $\mathrm{Tor}_2^A(k,k)$ is finite dimensional, while the converse is not true. There are k-algebras that are not finitely presented, with $\mathrm{Tor}_2^A(k,k)$ finite dimensional, just as there are groups with a finitely generated Schur multiplier which are not finitely presented.

The second Tor functor, $\mathrm{Tor}_2^A(k,k)$, has been studied extensively. Frohlich [5] gave a "Hopf formula" for it, Knus [6] gave another proof of the formula, and of the five term exact sequence, which also shows the connection to the group and Lie algebras cases, and recently Rosset [8] studied it from a different angle, in analogy to the Baer invariants of groups.

As mentioned above, Frohlich [5] gave a "Hopf formula" for the Schur multiplier of A. If $0 \to I \to R \to A \to 0$ is a presentation of A, where R is a free k-algebra, and δ is the augmentation ideal of R, then

$$\mathrm{Tor}_2^A(k,k) = (I \cap \delta^2)/(\delta I + I\delta).$$

Can Lewin's result be strengthened to the statement that if A is infinite dimensional then the Schur multiplier of R/I^2 is *not* finite dimensional? In other words, do the results of [2] (for groups) and [1] (for Lie algebras) extend to associative algebras? In this generality the question remains unanswered. In addition, the same question can be asked about R/I^n for $n \geq 2$. We shall prove that under the assumption that $A = R/I$ is an augmented domain, $\mathrm{Tor}_2^{R/I^n}(k,k)$ is in fact not finite dimensional.

THEOREM 1.1: *Let R be a finitely generated free associative algebra over a field k, and $I \subset \delta$ a non-zero two-sided ideal of R such that R/I is an infinite dimensional augmented integral domain over k. Then for $n > 1$,* $\mathrm{Tor}_2^{R/I^n}(k,k)$

is not finite dimensional, and hence I^n is not finitely generated as a two-sided ideal.

Note that the requirement that R/I be an integral domain is similar to arguments used in both the group and Lie algebra cases. In Lie algebras, the enveloping algebra (over a field) is always a domain, a fact that is used in [1]. In the case of groups, the group ring of $F/\gamma_n F$ is a domain, a fact that is used in [2].

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2. Proof

We shall need the following well known result.

THEOREM 2.1 (Cohn [4, Theorem II.4.3]): *Let R be a free algebra over a field k and I a left (resp. right) ideal of R. Then I is a free left (resp. right) R-module.*

In order to compute $\mathrm{Tor}_2^{R/I^n}(k,k)$ we can use Hopf's formula (note that for $n \geq 2$, $I^n \subset \delta^2$, thus $I^n \cap \delta^2 = I^n$)

$$\mathrm{Tor}_2^{R/I^n}(k,k) = I^n/(\delta I^n + I^n\delta) = k \otimes_R (I^n/I^{n+1}) \otimes_R k.$$

Any unadorned tensor is to be taken over k.

LEMMA 2.2: *Let R be a free algebra over a field k, and let I, J, K, L be any four two-sided ideals of R. Then*

$$I/KI \otimes_R J/JL \approx IJ/(KIJ + IJL).$$

Proof: By Theorem 2.1, I is a free right R module. Consider the mapping $I \otimes_R J \to I \otimes_R R \approx I$ which maps $i \otimes j \mapsto i \otimes j \mapsto ij$. Since I is flat, this is an injection. On the other hand, it is clear that the image of this mapping is IJ, so that $I \otimes_R J \approx IJ$. Now tensor this, over R, with R/K on the left, and R/L on the right. Thus

$$R/K \otimes_R I \otimes_R J \otimes_R R/L \approx R/K \otimes_R IJ \otimes_R R/L.$$

But the left hand side is isomorphic to $I/KI \otimes_R J/JL$ and the right hand side is isomorphic to $IJ/(KIJ + IJL)$. ∎

By an easy induction it follows from the previous lemma that $I^n/I^{n+1} \approx (I/I^2)^{\otimes_R^n}$, where this is to be taken to mean the n-fold tensor product over R. Therefore $\mathrm{Tor}_2^{R/I^n}(k,k) \approx k \otimes_R (I/I^2)^{\otimes_R^n} \otimes_R k$. However $k \otimes_R I/I^2 \approx I/\delta I$, and similarly $I/I^2 \otimes_R k \approx I/I\delta$. Note that the tensor products can be taken over A since I acts trivially on all these modules. Thus

$$\mathrm{Tor}_2^{R/I^n}(k,k) \approx I/\delta I \otimes_A (I/I^2)^{\otimes_A^{n-2}} \otimes_A I/I\delta.$$

A theorem of Lewin is now very useful.

THEOREM 2.3 (Lewin [7]): *Let R be a free algebra over a field k, with basis $X = \{x_1, \ldots, x_m\}$. Let U, V be two ideals of R and let T be the free $R/V - R/U$ bi-module with basis $\{t_1, \ldots, t_m\}$, then there is a bi-module monomorphism*

$$d : U \cap V/VU \to T.$$

Note that T is simply the direct sum of m copies of $R/V \otimes (R/U)^{op}$. These injections are the equivalent of the Magnus embeddings that are used to prove the analogous results for groups and Lie algebras in [2] and [1]. In fact, the injections are a kind of 'Universal Derivation', as defined in [3]. One way to understand the injections is to consider a mapping $U \cap V \to R/V \otimes_R J \otimes_R (R/U)^{op}$ where J is the kernel of the multiplication mapping $R \otimes R \to R$. The injection of the theorem sends $x \mapsto 1 \otimes (x \otimes 1 - 1 \otimes x) \otimes 1$.

Another way to understand the injection, as shown in [7], is as follows. Define a derivation $d : R \to T$ by declaring $d(1) = 0$ and $d(x_i) = t_i$. This defines d on R since it is required to be a k-linear derivation. Thus if $w = x_{i_1} \cdots x_{i_l}$ is a monomial of R then

$$d(w) = \sum_{j=1}^{l} (x_{i_1} \cdots x_{i_{j-1}} + V) t_{i_j} (x_{i_{j+1}} \cdots x_{i_l} + U).$$

It is shown in [7] that d, restricted to $U \cap V$, is a bi-module morphism, and that the kernel of d is VU.

This theorem will be used three times, to give us three bi-module injections. In the first case take $U = I$ and $V = \delta(R)$. The result is an injection $d_1 : I/\delta I \to T$ where T is the direct sum of m copies of $R/\delta \otimes A^{op} = A^{op}$, thus $T = (A^{op})^m$, a free right A module.

In the second case take $U = \delta$ and $V = I$ and the result is an injection $d_2 : I/I\delta \to T$, where $T = A^m$, a free left A module. A third case to consider is where $U = V = I$. In this case $T = (A \otimes A^{op})^m$, a free A bi-module, so the injection is $d_3 : I/I^2 \to (A \otimes A^{op})^m$.

The important fact is that in all three cases, since d_1, d_2, d_3 are injections, and the original bi-modules are non-zero (since $I \subset \delta$ is non-zero), then the mappings are non-zero.

LEMMA 2.4: *Let A be a domain over a field k, and F, resp. G, free right, resp. left, A modules, and let $f \in F$ and $g \in G$ be non-zero. Then the k-linear mapping $A \to F \otimes_A G$ given by $a \mapsto fa \otimes g$ is injective. In particular $f \otimes g \neq 0$.*

Proof: Suppose $\{\alpha_i\}$ is a basis of F as a right module and $\{\beta_j\}$ a basis of G as a left module. Since f and g are non-zero, then $f = \sum_i \alpha_i x_i$ and $g = \sum_j y_j \beta_j$ where $x_i, y_j \in A$, and there are indices i_0, j_0 such that x_{i_0} and y_{j_0} are non-zero. Since F and G are free modules then we have a right A-linear mapping $\mu : F \to A$ and a left A-linear mapping $\nu : G \to A$, such that $\mu(\alpha_i) = \delta_{ii_0}$ and $\nu(\beta_j) = \delta_{jj_0}$. This defines a mapping $\mu \otimes \nu : F \otimes_A G \to A \otimes_A A \approx A$, where the last isomorphism is through multiplication. If $a \in A$ then $a \mapsto \sum_{i,j} \alpha_i x_i a \otimes y_j \beta_j$. If $a \mapsto 0$ then $\mu \otimes \nu(\sum_{i,j} \alpha_i x_i a \otimes y_j \beta_j) = x_{i_0} a y_{j_0} = 0$. But since A is a domain, this means $a = 0$. ∎

LEMMA 2.5: *Let R be a finitely generated free associative algebra over a field k, and $I \subset \delta$ a non-zero two-sided ideal of R such that $A = R/I$ is an augmented domain. Then for any $l \geq 1$, there is a non-zero left A-linear mapping of $I^l/I^l\delta$ into a free left A module.*

Proof: For $l = 1$ this is obvious, just take $d_2 : I/I\delta \to A^m$. We proceed by induction on l. Let G_l be a free left A module, and $\phi_l : I^l/I^l\delta \to G_l$ a non-zero left A-linear mapping. We also have $d_3 : I/I^2 \to (A \otimes A^{op})^m$, which is a non-zero A bi-module mapping. This gives us a mapping $d_3 \otimes \phi_l : I/I^2 \otimes_A I^l/I^l\delta \to (A \otimes A^{op})^m \otimes_A G_l$. Set $G_{l+1} = (A \otimes A^{op})^m \otimes_A G_l$ and $\phi_{l+1} = d_3 \otimes \phi_l$. By Lemma 2.4, ϕ_{l+1} is a non-zero mapping, and since d_3 is left A-linear, so is ϕ_{l+1}. Since $(A \otimes A^{op})^m$ is a free bi-module, and G_l is a free left module, then G_{l+1} is a free left module. However, by Lemma 2.2, $I/I^2 \otimes_A I^l/I^l\delta = I/I^2 \otimes_R I^l/I^l\delta \approx I^{l+1}/I^{l+1}\delta$, so ϕ_{l+1} is the required mapping. ∎

THEOREM 2.6: *Let R be a finitely generated free associative algebra over a field k, and $I \subset \delta$ a non-zero two-sided ideal of R such that $A = R/I$ is an augmented domain. Then for $n \geq 2$ there is a k-linear injection of A into $\mathrm{Tor}_2^{R/I^n}(k,k)$. In particular, if A is infinite dimensional, then $\mathrm{Tor}_2^{R/I^n}(k,k)$ is also infinite dimensional.*

Proof: Consider

$$\mathrm{Tor}_2^{R/I^n}(k,k) \approx I/\delta I \otimes_A (I/I^2)^{\otimes_A^{n-2}} \otimes_A I/I\delta \approx I/\delta I \otimes_A I^{n-1}/I^{n-1}\delta.$$

Since $n - 1 \geq 1$, by Lemma 2.5 there is a left A-linear non-zero mapping $\phi : I^{n-1}/I^{n-1}\delta \to G$ where G is a free left A module. In addition, $d_1 : I/\delta I \to (A^{op})^m$ is a right A-linear non-zero mapping to a free right A module. Thus we can choose $x \in I/\delta I$, $y \in I^{n-1}/I^{n-1}\delta$ that do not map to zero. Consider the k-linear mapping, $\psi : A \to \mathrm{Tor}_2^{R/I^n}(k,k)$ defined by $\psi(a) = xa \otimes y$. Since by Lemma 2.4, $(d_1 \otimes \phi) \circ \psi$ is an injection, ψ must be an injection. ∎

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